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On Orthoptic and Isoptic Loci.

By Prof. Harold Hilton and Miss R. E. Colomb.

Section 1.

The nature of the orthoptic locus (the locus of the intersection of two perpendicular tangents) of a given real plane algebraic curve does not seem to have been investigated with any fulness, except that the degree of the orthoptic locus has been obtained in certain cases and a few of the more elementary properties of the locus have been given, at any rate when the given curve does not pass through the circular points at infinity.*

In the following discussion of the locus we shall denote the degree, class, number of nodes, number of cusps, number of bitangents, number of inflexions, deficiency of the given curve by n, m, δ, κ, τ, ι, D; and the corresponding quantities for the orthoptic locus by n', m', δ', κ', τ', ι', D'.

We take ω, ω' as the circular points at infinity. If we choose any points E, F on $\omega\omega'$ so that $(EF, \omega\omega')$ is harmonic, the orthoptic locus will be the locus of the intersection of tangents from E and F to the given curve.

It must be understood throughout that, when we discuss the orthoptic locus of some special type of curve, we imply that the curve is the most general curve of the special type discussed. Otherwise the orthoptic locus may degenerate or be simplified. For instance, the orthoptic locus of a central conic is in general a circle; but it becomes a pair of lines through ω and ω', if the conic divides $\omega\omega'$ harmonically.

It must be understood also that the line $\omega\omega'$ is never counted as part of the orthoptic locus.

Section 2.

First suppose (§§ 2–4) that the given curve has m^2 distinct ordinary foci; *i. e.*, no inflexional tangent or bitangent of the curve goes through ω or ω', the curve does not touch $\omega\omega'$ or go through ω or ω', etc.

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* Bassett, "Elementary Treatise on Cubic and Quartic Curves," §§ 68, 157; Zimmermann, *Crelle* 126 (1903), p. 183; Taylor, *Proc. Royal Soc.*, 37 (1884), p. 138, and *Messenger Math.*, 16 (1886), p. 1; Jonquières, *Nouv. Annales Math.*, 20 (1861); Loucher, *ibid.*, II, 11 (1892); Ljungh, Diss. "Ueber Isoptische und Orthoptische Kurven" (Lund, 1895). We have not been able to peruse this last paper.

Suppose two tangents are drawn to the curve from E and F meeting at P. Let E, and therefore F, approach ω. Then, if the tangents at E and F do not become consecutive, P will approach ω, the pencil $P(EF, \omega\omega')$ being harmonic. Hence, ultimately the orthoptic locus has one branch through ω for each pair of tangents from ω to the curve, and the tangent to the branch and the line $\omega\omega'$ form a harmonic pencil with the pair of tangents from ω to the curve.

Hence, through ω $\frac{1}{2}m(m-1)$ branches of the orthoptic locus pass, and similarly for ω', while there is no other point of the orthoptic locus on $\omega\omega'$. Hence, $n' = m(m-1)$.

Moreover, if ωH, $\omega' H$ and ωK, $\omega' K$ are two conjugate pairs of tangents from ω and ω' (so that H, K are real foci) and C is the middle point of HK, it follows from the harmonic properties of a quadrilateral that $\omega(C\omega', HK)$ and $\omega'(C\omega, HK)$ are harmonic pencils. Therefore, $C\omega$ and $C\omega'$ touch the orthoptic locus at ω and ω', *i. e.*, C is a *singular focus* of the orthoptic locus. Hence,

The orthoptic locus of a curve of class m is in general of degree $m(m-1)$, *and its singular foci are the* $\frac{1}{2}m(m-1)$ *middle points of the segments joining any two of the m real foci of the given curve.*

The intersections of a curve with its orthoptic locus are given by the following theorem, the proof of which we leave to the reader:

A curve of degree n and class m in general cuts its orthoptic locus at the $2m$ *points of contact of tangents from the circular points to the curve, touches it at the feet of the* $m(m+n-4)$ *normals which are also tangents to the curve, and cuts it in the* $m(m-3)(n-2)$ *points from which two perpendicular tangents can be drawn to the curve neither of which coincides with the tangent at the point.*

Section 3.

We now consider the *class* of the orthoptic locus. It will be sufficient to find the number of tangents which can be drawn from ω' to the locus.

First consider the tangents from ω' to the locus whose points of contact are not on $\omega\omega'$.

Suppose PH, $P'H$ to be two consecutive tangents to the curve and PK, $P'K$ the consecutive tangents perpendicular to PH, $P'H$. Suppose, moreover, that PP' passes through ω'. Then, because $P(HK, \omega\omega')$ and $P'(HK, \omega\omega')$ are harmonic, HK passes through ω. But H and K are ultimately the points of contact of the tangents PH and PK. Hence,

If a tangent from ω' *to the orthoptic locus touches it at a finite point P, the line joining the points of contact of the two perpendicular tangents from P to the curve passes through* ω.

Now suppose any line whatever through ω meets the given curve in H and K. Let the tangents at H and K meet at T, and consider the envelope of the line TV where $T(HK, \omega V)$ is harmonic. By what has just been said, every tangent from ω' to the envelope which does not coincide with $\omega\omega'$ will be a tangent to the orthoptic locus. Now, in general, the envelope will not touch $\omega\omega'$, and hence the number of tangents from ω' to the orthoptic locus with a finite point of contact is equal to the class of the envelope.

To find the class of the envelope, we find the number of tangents from ω to the curve. The line TV can not pass through ω unless ωHK touches the given curve at H but not at K (or vice versa). If this is the case, however, ωHK will be a tangent to the envelope at a point U such that $(\omega U, HK)$ is harmonic. Hence, each tangent from ω to the curve is a $(n-2)$-ple tangent to the envelope; so that the class of the envelope is $m(n-2)$.

Now the only tangents from ω' to the orthoptic locus with their points of contact on $\omega\omega'$ are the $m(m-1)$ tangents at the $\frac{1}{2}m(m-1)$-ple point ω' itself. Hence,

$$m' = m(n-2) + m(m-1) = m(m+n-3).$$

Section 4.

We now consider the double points and cusps of the orthoptic locus.

Let PH, $P'H$ be two consecutive tangents to the given curve and PK, $P'K$ the consecutive tangents perpendicular to PH, $P'H$. Then P, P', H, K are concyclic, and therefore the angle $KPP'=$ the angle KHP'. Hence in the limit:

If PH, PK are two perpendicular tangents to a curve, the angle between PK and the tangent to the orthoptic locus at P is equal to the angle KHP.

If PH is a bitangent to the given curve, there are two points of contact H on PH, and therefore there are two tangents to the orthoptic locus at P; *i. e.*, P is a double point of the orthoptic locus. It will be readily seen that, if PH is an inflexional tangent, P is a cusp of the orthoptic locus. Hence,

The orthoptic locus has a double point at each of the $m\tau$ intersections of a bitangent of the given curve with a perpendicular tangent, and has a cusp at each of the $m\iota$ intersections of an inflexional tangent of the given curve with a perpendicular tangent.

It will readily be seen that all the cusps of the orthoptic locus are given in this way. Hence,

$$\varkappa' = m\iota.$$

We have now found n', m', $\varkappa'$ and can deduce δ', τ', ι' from Plücker's equations. We find

$$n'=m(m-1),\quad m'=m(m+n-3),\quad \delta'=\tfrac{1}{2}m\{(m+1)(m-2)^2+2\tau\},$$
$$\varkappa'=m\iota,\quad \tau'=\tfrac{1}{2}m\{m(m+n)^2-(6m^2+6mn+n^2)-m+22+2\delta\},$$
$$\iota'=m(3m+\varkappa-6),\quad D'=\tfrac{1}{2}(m-1)(m-2)+mD.$$

Of the $\frac{1}{2}m\{(m+1)(m-2)^2+2\tau\}$ nodes of the orthoptic locus $m\tau$ are the intersections of the bitangents of the given curve with perpendicular tangents, and $\frac{1}{4}m(m+1)(m-1)(m-2)$ are accounted for by the $\frac{1}{2}m(m-1)$-ple points of the orthoptic locus at ω and ω'. There remain $\frac{1}{4}m(m+1)(m-2)(m-3)$ others. Hence,

There are in general $\frac{1}{4}m(m+1)(m-2)(m-3)$ points from which two pairs of mutually perpendicular tangents can be drawn to a given curve of class m.

The results of §§ 2–4 are illustrated by the conic whose orthoptic locus is a circle through the intersection of the conic and the directrices.

Section 5.

In §§ 5–7 we give very briefly without proof the properties of the orthoptic locus of curves bearing some special relation to the line $\omega\omega'$.

If the given curve touches $\omega\omega'$ at Y, the orthoptic locus has a branch through Z, where $(YZ, \omega\omega')$ is harmonic, corresponding to each tangent from Z to the given curve other than $\omega\omega'$.

We find that, if the given curve touches $\omega\omega'$ at k points (which may or may not be cusps of the curve),

$$n'=(m-k)(m-1),\quad m'=(m-k)(m+n-3-k),\quad \varkappa'=(m-k)\iota.$$

If the given curve has $\omega\omega'$ as inflexional tangent at Y, the branches of the orthoptic locus through Z are all linear and touch $\omega\omega'$ at Z (more generally, if the curve has r-point contact with $\omega\omega'$ at Y, each branch of the orthoptic locus has $(r-1)$-point contact with $\omega\omega'$ at Z), and that

$$n'=(m-1)(m-2),\quad m'=(m-2)(m+n-4),\quad \varkappa'=(m-2)(\iota-1).$$

As examples of the results of §§ 2–5 consider

(i) A parabola, whose orthoptic locus is its directrix.

(ii) $3(x+y)=x^3$, whose orthoptic locus is

$$81y^2(x^2+y^2)-36(x^2-2xy+5y^2)+128=0.$$

Here we have

$$n=3,\quad m=3,\quad \delta=0,\quad \varkappa=1,\quad \tau=0,\quad \iota=1,$$
$$n'=4,\quad m'=4,\quad \delta'=2,\quad \varkappa'=2,\quad \tau'=1,\quad \iota'=2.$$

The given curve touches $\omega\omega'$ at its cusp $(0, \infty)$, and therefore the orthoptic locus has a double point at $(\infty, 0)$ the tangents there being $9y^2=4$, which are also tangents to the given curve.*

The intersections $(\mp\frac{2}{3}\sqrt{2}, \pm\frac{2}{3}\sqrt{2})$ of the inflexional tangent of the given curve with the perpendicular tangents are cusps of the orthoptic locus.

(iii) $y^2=x^3$, whose orthoptic locus is $729y^2=108x-16$.

The curve has $\omega\omega'$ as inflexional tangent at $(0, \infty)$, and therefore the orthoptic locus touches $\omega\omega'$ at $(\infty, 0)$.

The orthoptic locus meets the curve where $x=-4/9$, which gives the points of contact of the tangents from ω, ω' to the curve. The orthoptic locus touches the curve where $x=2/9$, *i. e.*, at the feet of the two normals of the curve which are also tangents.

Section 6.

In § 5 we considered a curve for which $\omega\omega'$ was a multiple tangent, tacitly assuming that no two of the points of contact divided $\omega\omega'$ harmonically.

If this were the case, the branches of the orthoptic locus at ω and ω' are as in § 5, but the nature of the other points of the locus on $\omega\omega'$ is altered, and the degree of the locus is in general lowered. We shall suppose that $\omega\omega$ touches the curve at Y and Z only, so that $(\omega\omega', YZ)$ is harmonic.

(A) Let the curve have ordinary contact at Y and Z. The locus has $(m-3)$ branches through Y due to two perpendicular tangents to the curve, one of which touches near Z and the other passes close to Y but does not touch near Y or Z. Similarly, it has $(m-3)$ branches through Z. It has also in addition a branch cutting $\omega\omega'$ due to two perpendicular tangents, one of which touches near Y and the other near Z.

Example. The tricuspidal quartic $x^2y^2-4(x^3+y^3)+18xy-27=0$ with orthoptic locus $x+y+2=0$.

(B) Let the curve have ordinary contact at Y and a cusp at Z.

The locus has $(m-3)$ branches through Y and $(m-4)$ through Z, and in addition another branch touching $\omega\omega'$ at Z.

(C) Let the curve have ordinary contact at Y and an inflexion at Z.

The locus has $(m-4)$ branches touching $\omega\omega'$ at Y and $(m-4)$ branches through Z, and in addition a branch touching $\omega\omega'$ at Y.

Example. The curve with tangential equation $\lambda^4+2\mu^4=\lambda^2\mu$ and orthoptic locus $2x^2+y+3=0$.

(D) Let the curve have inflexions at Y and Z.

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* Because $\omega\omega'$ is a *cuspidal* tangent of the given curve.

The locus has $(m-5)$ branches touching $\omega\omega'$ at Y and similarly for Z. It has in addition a branch touching $\omega\omega'$ elsewhere.

Example. $\lambda^2\mu^2=\lambda^5+\mu^5$ with orthoptic locus

$$(x-y)^2+2(x+y)+4=0.$$

and so on in all similar cases.

Section 7.

We now notice briefly some cases in which the given curve passes through ω and ω'.

If the curve has a linear branch through ω and ω', the orthoptic locus has a branch through ω touching the curve at ω, and a superlinear branch of order 2 at ω for each of the $m-2$ tangents from ω to the curve not touching at ω. The line $\omega\omega'$ and the tangent to the superlinear branch form a harmonic pencil with the tangent at ω and the tangent from ω to the given curve. The orthoptic locus has also a linear branch through ω corresponding to each of the $\frac{1}{2}(m-2)(m-3)$ pairs of tangents from ω to the given curve not touching at ω (see §2). We find

$$n'=m(m-1), \quad m'=m(m+n-5)+4, \quad \varkappa'=2(m-2)+m\iota.$$

The reader may illustrate the above statements on the circle whose orthoptic locus is a concentric circle, or on the cissoid $(x+1)^3+xy^2=0$ whose orthoptic locus is

$$16(x^2+y^2)^3+108(x^2+y^2)(x^2+2y^2)+108x^3+729y^2=0.$$

If the given curve has two linear branches through ω, each branch contributes its quota to the form of the orthoptic locus at ω as stated above. The orthoptic locus has also two superlinear branches of order 2 given by the intersections of two tangents to the curve, one of which touches one branch and the other touches the other branch near ω. The two superlinear branches have a common tangent, and this tangent and $\omega\omega'$ form a harmonic pencil with the tangents to the curve at ω.

Similarly, if the given curve has more than two branches through ω.

If the curve touches $\omega\omega'$ at ω and ω', the orthoptic locus has $m-2$ linear branches through ω, of which $m-3$ touch $\omega\omega'$ at ω in addition to the $\frac{1}{2}(m-3)(m-4)$ branches through ω obtained as in § 2, so that

$$n'=2+4(m-3)+(m-3)(m-4)=(m-1)(m-2).$$

Example. The three-cusped hypocycloid whose orthoptic locus is a circle.

The reader may consider the case in which the given curve has cusps at ω and ω', and illustrate by the cardiode whose orthoptic locus consists of a circle and a limaçon, all three curves having a common singular focus.

Section 8.

If an inflexional or multiple tangent of the given curve passes through ω, this tangent is part of the orthoptic locus. In this section n', m', ... refer to the remainder of the orthoptic locus excluding such tangents.

If a bitangent passes through ω, the orthoptic locus has $2(m-2)$ linear branches through ω touching each other in pairs, and $\frac{1}{2}(m-2)(m-3)$ other linear branches through ω. We have

$$n'=(m+1)(m-2), \quad m'=m(m+n-3)-4, \quad \varkappa'=m\iota.$$

If an inflexional tangent passes through ω, the orthoptic locus has $(m-2)$ superlinear branches of order 2 at ω, and $\frac{1}{2}(m-2)(m-3)$ other linear branches through ω. We have

$$n'=(m+1)(m-2), \quad m'=m(m+n-4), \quad \varkappa'=m\iota-4.$$

Example. $(2x+1)^3=27(x^2+y^2)$ with orthoptic locus

$$(16y^4-16xy^2+4x^2+4y^2-4x-1)(x^2+y^2)=0.$$

If the curve has an inflexion at ω and ω', the inflexional tangents count twice over as part of the locus, and $n'=m(m-1)-4$.

Section 9.

The nature of the *locus of the intersection of two perpendicular normals* of a given curve is readily obtained, for it is the orthoptic locus of its evolute.

Taking the general curve of § 2, its evolute is of degree $n_1=3m+\varkappa$, is of class $m_1=n+m$, has $\iota_1=0$ inflexions, etc. The evolute has n cusps at which $\omega\omega'$ is the tangent.

Hence, putting n_1, m_1, ... for n, m, ... in § 5, and putting $k=n$, we see that the locus of the intersection of two perpendicular normals is of degree $n_1'=m(m+n-1)$, is of class $m_1'=m(4m+\varkappa-3)$, and has $\varkappa_1'=0$ cusps.

If the given curve touches $\omega\omega'$,

$$n_1'=(m-1)(m+n-2), \quad m_1'=(m-1)(4m+\varkappa-6), \quad \varkappa_1'=0.$$

For example, the evolute of a parabola is a semi-cubical parabola whose orthoptic locus is a parabola.

If the given curve has $\omega\omega'$ as inflexional tangent,

$$n_1'=(m-2)(m+n=3), \quad m_1'=(m-2)(4m+\varkappa-9), \quad \varkappa_1'=0.$$

The reader may illustrate on the semi-cubical parabola.

Section 10.

We here enumerate the types of real curve whose orthoptic locus is of the first or second degree.

I. *Orthoptic locus a straight line.*
   (i) Parabola.
   (ii) $n=4$, $m=3$. Curve touches $\omega\omega'$ in two points dividing $\omega\omega'$ harmonically.

II. *Orthoptic locus a circle.*
   (i) Circle.
   (ii) Conic.
   (iii) $n=4$, $m=3$. Curve touches $\omega\omega'$ at ω and ω'.

III. *Orthoptic locus a parabola.*
   (i) $n=3$, $m=3$. Curve has $\omega\omega'$ as inflexional tangent.
   (ii) $n=5$, $m=4$. Curve has $\omega\omega'$ as an ordinary and as an inflexional tangent, the points of contact dividing $\omega\omega'$ harmonically.
   (iii) $n=6$, $m=5$. Curve has two inflexions dividing $\omega\omega'$ harmonically, at which $\omega\omega'$ is the tangent.

IV. *Orthoptic locus a conic.*
   (i) $n=4$, $m=3$. $\omega\omega'$ is a bitangent.
   (ii) $n=6$, $m=4$. $\omega\omega'$ is a triple tangent, two of the points of contact dividing $\omega\omega'$ harmonically.
   (iii) $n=8$, $m=5$. $\omega\omega'$ is a quadruple tangent, two pairs of points of contact dividing $\omega\omega'$ harmonically.

As an example the reader may enumerate the fourteen types of curve whose orthoptic locus is a cubic, and the thirty-eight types whose orthoptic locus is a quartic.

Section 11.

We now state the properties of the *isoptic locus* of a given curve; *i. e.*, the locus of the intersection of two tangents inclined at any given angle α.*

To each pair of tangents to the curve from ω correspond two branches of the isoptic locus through ω. We have

$$n'=2m(m-1), \quad m'=2m(n+m-2), \quad \varkappa'=2m\iota.$$

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* We suppose $\alpha \not\equiv 0$ and $\alpha \not\equiv \frac{1}{2}\pi$. It is not in general possible to distinguish the case in which two tangents cut at an angle α, from that in which they cut at an angle $\pi-\alpha$. But consider the case of the circle or parabola.

The curve cuts the isoptic locus at the $2m(m-3)(n-2)$ points from which two tangents can be drawn inclined at an angle α and neither of them touching at the point. It touches the locus at the $2m$ points of contact of tangents from ω and ω' * and at the $2m(m+n-4)$ points such that a line making an angle α with the tangent at the point touches the curve elsewhere.

If the curve touches $\omega\omega'$ at Y, the locus has $(m-1)(m-2)$-ple points at ω and ω', and has an $(m-1)$-ple point at Z where $(Y\omega\, Z\omega')$ has the cross-ratio $e^{\pm 2i\alpha}$. We find

$$n' = 2(m-1)^2, \quad m' = 2(m-1)(m+n-3), \quad \varkappa' = 2(m-1)\iota.$$

If the curve has $\omega\omega'$ as inflexional tangent at Y, the locus has $(m-2)(m-3)$-ple points at ω and ω' and has $(m-2)$ linear branches touching $\omega\omega'$ at Z. We find

$$n' = 2(m-1)(m-2), \quad m' = 2(m-2)(m+n-3), \quad \varkappa' = 2(m-2)(\iota-1).$$

As examples consider (we write k for $\tan^2\alpha$):

(i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with isoptic locus $k(x^2+y^2-a^2-b^2)^2 = 4a^2b^2\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1\right)$.

(ii) $y^2 = 4ax$ with isoptic locus $y^2 - 4ax = k(x+a)^2$.

(iii) $y^2 = x^3$ with isoptic locus

$$3^{12}k(x^2-ky^2)^2 - 3^9 2^3\{(k^2+k+2)x^3 + (k^3+3k^2)xy^2\}$$
$$+ 3^6 2^4\{(k^3+4k^2+9k)x^2 + (2k^3+27k^2+54k+27)y^2\} - 3^3 2^7(k^3+3k^2)x + 2^8k^3 = 0,$$

touching the curve where $x = -4/9$ and $k(9x-2)^2 = 81x$, and touching $\omega\omega'$ where $x^2 = ky^2$.

To find the nature of the locus of the intersection of two tangents, one drawn to each of two given curves and inclined at a constant angle, we have only to take away the isoptic locus of each curve considered separately from the isoptic locus of the (degenerate) curve, consisting of the two curves taken together.†

The case in which the given curves are both circles is well known.

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* Contrast with the orthoptic locus.

† Taylor, *loc. cit.*